

# Rigorous and Approximated Solutions of the Consolidation Problem for a Soil Layer with Finite Thickness under Cyclic Mechanical Loading

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## Formulation of the problem

In poroelasticity there are two important assumptions, namely (see [2] and [6]):

- (i) the stress-strain relation for the solid matrix follows the Hooke's law for isotropic linear elastic media,
- (ii) Darcy's law governs the fluid flow within the pore system.

Using these assumptions it can be shown that the governing equation for one-dimensional (vertical) consolidation due to the time dependent mechanical loading  $L(t)$ , see [6], is governed by the following equation:

$$(1) \quad \frac{\partial u}{\partial t} = C_z \frac{\partial^2 u}{\partial z^2} + \eta \frac{dL}{dt},$$

where  $u(z, t)$  is the excess pore water pressure at depth  $z$  and time  $t$ ,  $C_z$  is the coefficient of consolidation in vertical direction  $z$  (Fig. 1) and  $\eta = \frac{\alpha}{\alpha + n\beta}$ . Here  $\alpha$ ,  $\beta$  and  $n$  are the compressibility of the solid, the compressibility of the water and the porosity, respectively. In case water is considered to be incompressible,  $\eta$  equals one. The coefficient of consolidation is related to the mechanical and hydraulic characteristics of the porous medium (e.g. soil) as follows:

$$(2) \quad C_z = \frac{k K_s}{\gamma \left(1 + \frac{n\beta}{K_s}\right)},$$

where  $K_s$  is the bulk modulus of the solid phase,  $k$  is the hydraulic permeability and  $\gamma$  represents the fluid volumetric weight.

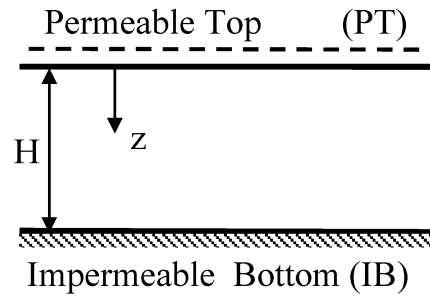


Fig. 1. Problem geometry

The boundary and the initial conditions are defined as follows (according to the notations in Fig. 1)

$$(3) \quad u(0, t) = 0, \quad \frac{\partial u}{\partial z}(H, t) = 0,$$

$$(4) \quad u(z, 0) = 0.$$

More information about the geotechnical applications and experimental data where the above formulated problem has place can be found in [5] and [4]. The task is to derive an analytical solution to the above formulated boundary value problem. Next, based on the analytical solution to evaluate the following sub-tasks:

- explicitly derive the phase shift between excess pore water pressure  $u(z, t)$  and the applied load with time (for a fixed depth) especially at the bottom ( $z = H, t \rightarrow \infty$ ), the phase shift or lag is a positive or negative delay of the excess pore water pressure as compared to the applied surface load that may vary with depth;
- parameter analysis for the solution regarding permeability  $k$  and as a function of relevant parameters (stratum depth  $H$ , bulk modulus  $K_s$ , phase shift, load amplitude  $q$  and loading period  $d$ );
- parameter analysis for the phase shift  $\psi$  as a function of the fluid and solid phase compressibility and soil permeability.

### Analytical solution

In this section our task is to obtain an analytical solution for the above formulated problem. In order to do that we will first formulate the equivalent homogeneous form of Eq. (1), see e.g. [3]. For that purpose we introduce a new variable:

$$(5) \quad v(z, t) = u(z, t) - \eta \int_0^t \dot{L}(\tau) d\tau,$$

where the load function is given by

$$(6) \quad L(t) = q \sin^2 \left( \frac{\pi}{d} t \right) = \frac{q}{2} - \frac{q}{2} \cos \left( \frac{2\pi}{d} t \right).$$

Therefore Eq. (5) takes the form

$$(7) \quad v(z, t) = u(z, t) - \eta \frac{q}{2} \cos \left( \frac{2\pi}{d} t \right).$$

In the new variable  $v$  our problem is formulated as:

$$(8) \quad \frac{\partial v}{\partial t} = C_z \frac{\partial^2 v}{\partial z^2},$$

with boundary conditions:

$$(9) \quad v(0, t) = \eta \frac{q}{2} \cos \left( \frac{2\pi}{d} t \right), \quad \frac{\partial v}{\partial z}(H, t) = 0.$$

The boundary condition at the bottom of the layer ( $z = H$ ) can be expressed as  $v(H, t) = C$ , where  $C$  is some constant, independent of  $z$  and  $t$ . If we assume that  $C = 0$ , the boundary condition at the bottom is then given as

$$(10) \quad v(H, t) = 0.$$

We seek for a harmonic solution of the homogeneous problem of the form

$$(11) \quad v(z, t) = V(z) e^{i \frac{2\pi}{d} t},$$

where the function  $V(z)$  is obtained from separation of variables and thus becomes

$$(12) \quad V(z) = C_1 e^{\frac{1+i}{\sqrt{dC_z/\pi}} z} + C_2 e^{-\frac{1+i}{\sqrt{dC_z/\pi}} z}.$$

In order to obtain Eq. (12) in the form that is given, we have used the expression  $\sqrt{i} = (1 + i)/\sqrt{2}$ . Since the solution that we seek for is a complex function, we introduce an additional complex disturbance at  $z = 0$  and hence it becomes

$$(13) \quad v(0, t) = \eta \frac{q}{2} \left[ \cos \left( \frac{2\pi}{d} t \right) + i \sin \left( \frac{2\pi}{d} t \right) \right] = \eta \frac{q}{2} e^{i \frac{2\pi}{d} t}.$$

This allows us to determine the constant in Eq. (12). The solution for the variable  $v$  takes the final form:

$$(14) \quad v(z, t) = \left[ C_1 e^{\frac{1+i}{\sqrt{dC_z/\pi}} z} + C_2 e^{-\frac{1+i}{\sqrt{dC_z/\pi}} z} \right] e^{i \frac{2\pi}{d} t},$$

with

$$(15) \quad C_1 = -\eta \frac{q}{2} \frac{e^{-\frac{1+i}{\sqrt{dC_z/\pi}} H}}{e^{\frac{1+i}{\sqrt{dC_z/\pi}} H} - e^{-\frac{1+i}{\sqrt{dC_z/\pi}} H}},$$

and

$$(16) \quad C_2 = \eta \frac{q}{2} + \eta \frac{q}{2} \frac{e^{-\frac{1+i}{\sqrt{dC_z/\pi}} H}}{e^{\frac{1+i}{\sqrt{dC_z/\pi}} H} - e^{-\frac{1+i}{\sqrt{dC_z/\pi}} H}}.$$

We could proceed further with obtaining the real part of the above expression, but due to the complexity of the expressed solution we follow the approach of [1]. Therefore we assume that the relation  $H \gg \sqrt{dC_z/\pi}$  holds, which allows us to neglect the term with  $C_1$  in Eq. (12) and hence

$$(17) \quad Re(v(z, t)) = \eta \frac{q}{2} e^{-\frac{z}{\sqrt{dC_z/\pi}}} \cos \left( \frac{2\pi}{d} t - \frac{z}{\sqrt{dC_z/\pi}} \right).$$

The real part in Eq. (17) is the one corresponding to the real part of the disturbance, therefore transforming the expression for the variable  $u$  yields

$$(18) \quad u(z, t) = \eta \frac{q}{2} e^{-\frac{z}{\sqrt{dC_z/\pi}}} \cos \left( \frac{2\pi}{d} t - \frac{z}{\sqrt{dC_z/\pi}} \right) + \eta \frac{q}{2} \cos \left( \frac{2\pi}{d} t \right).$$

Using this analytical solution we can have an explicit expression for the phase shift  $\psi(z)$  between the excess pore water pressure and the applied load:

$$(19) \quad \psi(z) = \arctan \left[ \frac{e^{-\frac{z}{\sqrt{dC_z/\pi}}} \sin \left( -\frac{z}{\sqrt{dC_z/\pi}} \right)}{1 - e^{-\frac{z}{\sqrt{dC_z/\pi}}} \cos \left( -\frac{z}{\sqrt{dC_z/\pi}} \right)} \right].$$

Thus we conclude that when  $H \gg \sqrt{dC_z/\pi}$  holds the phase shift depends only on the period of the applied load and the consolidation coefficient.

### Numerical solution

In this section we will find the numerical solution of our problem. The partial differential equation (1) can be written in the following form:

$$(20) \quad \frac{\partial u}{\partial t} = C_z \frac{\partial^2 u}{\partial z^2} + A \sin\left(\frac{2\pi t}{d}\right),$$

and the boundary and the initial conditions are given by Eqs. (3) and (4). The equation is discretized by the finite element method. As a result, a system of ordinary differential equations (ODE) is obtained. Because the system is linear and the time dependent mechanical loading is harmonic, the steady-state response of the system will be also harmonic. Thus, the vector of generalized coordinates is expressed by harmonic function and the system of ODE is transformed into an algebraic linear system.

Equation (1) is written in variational form, integration by parts is performed and taking into account the boundary conditions, the following equation is obtained:

$$(21) \quad \int_{\Omega} \frac{\partial u(z, t)}{\partial t} v(z) dz + C_z \int_{\Omega} \frac{\partial u(z, t)}{\partial z} \frac{\partial v(z)}{\partial z} dz = A \sin\left(\frac{2\pi t}{d}\right) \int_{\Omega} v(z) dz,$$

where  $v(z)$  are the trial functions and  $\Omega \in [0, H]$ .

Following the standard FEM approach, the pressure  $u(z, t)$  is approximated by shape functions and generalized coordinates in a local coordinate system:

$$(22) \quad u_h(\xi, t) = \sum_{i=1}^n N_i(\xi) q_i(t) \in \mathcal{V}_h,$$

where  $\xi$  is the local coordinate given by  $\xi = \frac{z}{H}$  and  $n$  is the number of shape functions used in the model. In the current case, one element is used, and accuracy is achieved by adding higher order polynomials. This approach is also known as  $p$ -FEM. The shape functions  $N_i(\xi)$  have to satisfy the essential boundary conditions. They are given in Figure 2. Here  $q_i(t)$  are the generalized coordinates.

Switching from  $\Omega$  to the reference interval  $[0, 1]$  and replacing the expression for  $u_h(\xi, t)$  into Eq. (21) the following system of first order ODE is obtained:

$$(23) \quad \mathbf{M}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{F} \sin\left(\frac{2\pi t}{d}\right),$$

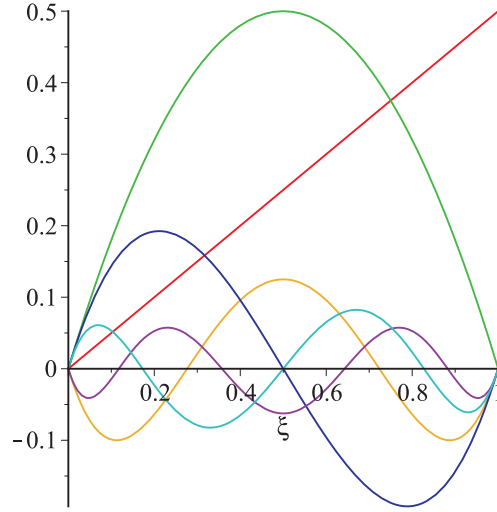


Fig. 2. Shape functions for the pressure.

where  $\mathbf{M}$  is the mass matrix,  $\mathbf{K}$  is the stiffness matrix,  $\mathbf{F}$  is the vector of generalized external forces and  $\mathbf{q}(t)$  is the vector of generalized coordinates. The matrices have the following form:

$$\begin{aligned}\mathbf{M} &= H \int_0^1 \mathbf{N}(\xi) \mathbf{N}(\xi)^T d\xi, \\ \mathbf{K} &= C_z \frac{1}{H} \int_0^1 \frac{d\mathbf{N}(\xi)}{d\xi} \frac{d\mathbf{N}(\xi)^T}{d\xi} d\xi, \\ \mathbf{F} &= A H \int_0^1 \mathbf{N}(\xi) d\xi,\end{aligned}$$

where  $\mathbf{N}(\xi)$  is the vector of shape functions.

Taking into account that the system is linear with harmonic excitation, the steady-state response is known to be harmonic function. Thus, the vector of generalized coordinates is expressed in the following way:

$$(24) \quad \mathbf{q}(t) = \mathbf{a} \cos\left(\frac{2\pi t}{d}\right) + \mathbf{b} \sin\left(\frac{2\pi t}{d}\right),$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are vectors with the same size as  $\mathbf{q}$ . This expression for  $\mathbf{q}$  is exact for the steady-state response of the linear system. If one wants to study

the transient response of the system, a time integration method, such as Runge-Kutta should be used. Substituting (24) into the system (23), the following algebraic system is obtained:

$$(25) \quad \left( \frac{2\pi}{d} \begin{bmatrix} \mathbf{0} & \mathbf{M} \\ -\mathbf{M} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{bmatrix} \right) \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{F} \end{Bmatrix}.$$

The vectors  $\mathbf{a}$  and  $\mathbf{b}$  are computed from this equation. These vectors determine the shape of the solution related with cos and sin terms:

$$(26) \quad u_{\cos}(\xi, t) = \sum_{i=1}^n N_i(\xi) a_i,$$

$$(27) \quad u_{\sin}(\xi, t) = \sum_{i=1}^n N_i(\xi) b_i.$$

The solution  $u_h(\xi, t)$  is obtained to be (in local coordinate system):

$$(28) \quad u_h(\xi, t) = u_{\cos}(\xi, t) \cos\left(\frac{2\pi t}{d}\right) + u_{\sin}(\xi, t) \sin\left(\frac{2\pi t}{d}\right) = \sqrt{u_{\cos}(\xi, t)^2 + u_{\sin}(\xi, t)^2} \cos\left(\frac{2\pi t}{d} - \psi(\xi)\right),$$

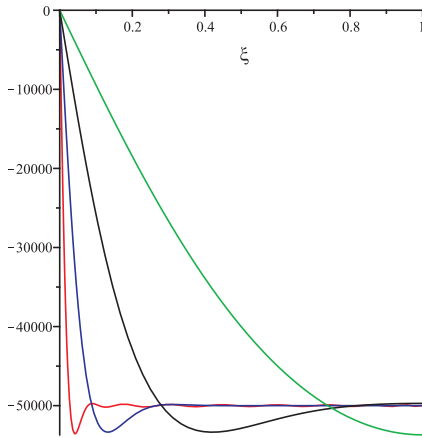


Fig. 3.  $u_h(\xi, 0)$  for different values of  $C_z$ , red -  $C_z = 10^{-8}$ , blue -  $C_z = 10^{-7}$ , black -  $C_z = 10^{-6}$ , green -  $C_z = 10^{-5}$

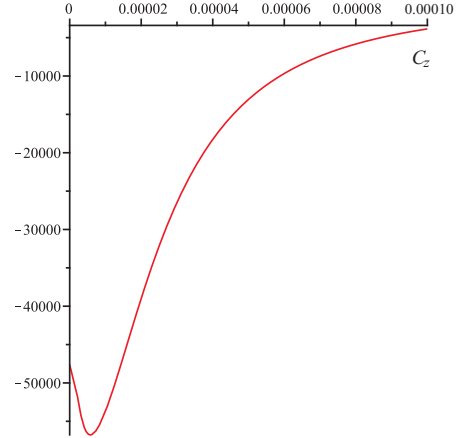


Fig. 4.  $u_h(\xi, 0)$  as a function of  $C_z$  for  $\xi = 1$  ( $z = H$ )

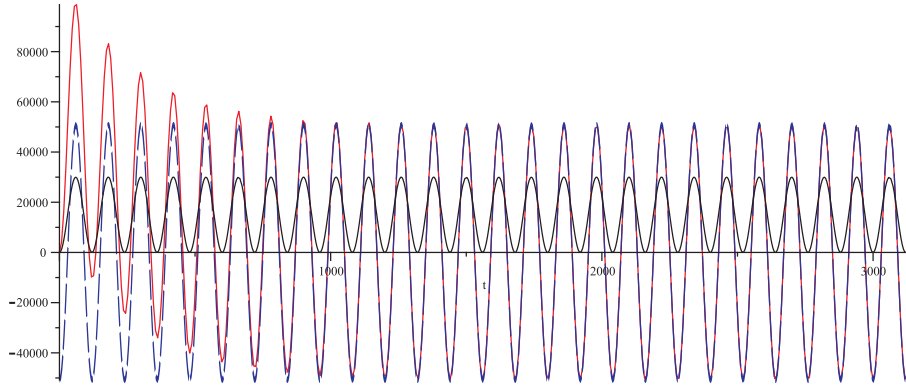


Fig. 5.  $u_h(\xi, t)$  in time domain for  $\xi = 1$ , red – Runge-Kutta method with zero initial conditions, blue – steady-state response of the FEM, black – applied load (amplitude is increased in order to be visible in the plot)

where  $\psi(\xi)$  is the phase expressed by:

$$(29) \quad \psi(\xi) = \arctan \left( \frac{u_{\sin}(\xi, t)}{u_{\cos}(\xi, t)} \right).$$

The result given for  $u_h$  in Eq. 28 is visualised in Figures 3 and 4. The comparison between the steady state and transient numerical solutions in time domain is presented in Fig. 5.

### Phase shift

The explicit expression for the phase shift  $\psi(z)$  as given in (19) is analyzed first. The plot for  $\psi$  as a function of the consolidation coefficient  $C_z$  is shown on Fig. 6. One can notice a drop in the phase shift when  $C_z \approx 0$  and an increase after that drop. At  $C_z \approx 5 \times 10^{-6}$  the slope of the graph decreases and  $\psi$  tends asymptotically to  $\pi/4$ .

Let us analyze in more detail the drop in the graph, i.e. negative phase shift. In Fig. 7 it is shown again a plot of  $\psi$  as a function of  $C_z$ , but now it is magnified so that one can see closely the interval with negative phase shift. From the figure we conclude that in the interval between  $1.1 \times 10^{-9}$  and  $3.35 \times 10^{-7}$  for  $C_z$  there is a negative phase shift between the excess pore water pressure and the applied load. This means that after consolidation has taken place the response from the system can be before the actual load for specific parameters. The phase shift as a function of  $C_z$  determined based on the FEM solution is given in Fig. 8.



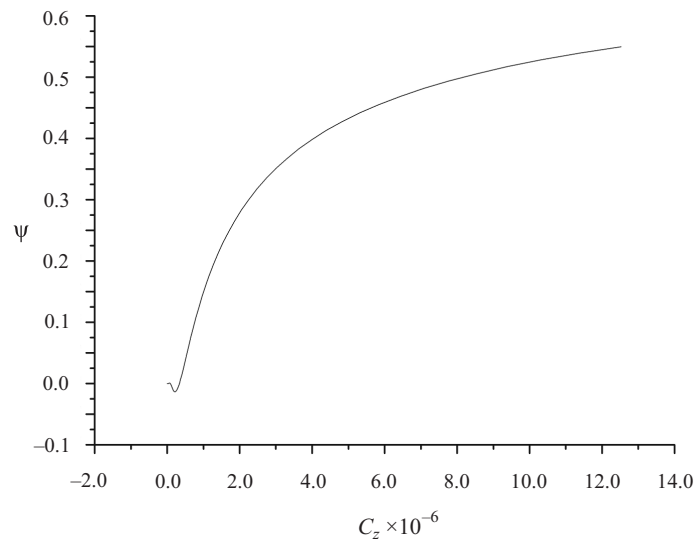


Fig. 6. Phase shift as a function of the consolidation coefficient

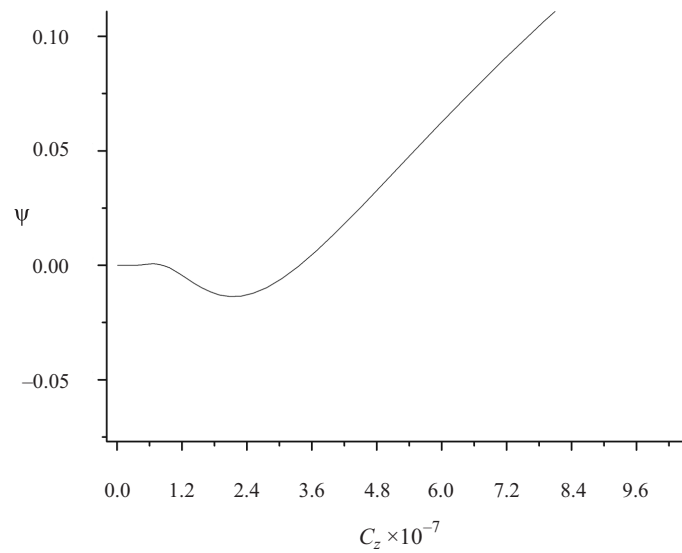


Fig. 7. Interval with negative phase shift in detail

### Closing Remarks

We have shown that the analytical solution of equation (1) under reasonable assumption on the relation between frequency, permeability and layer thickness

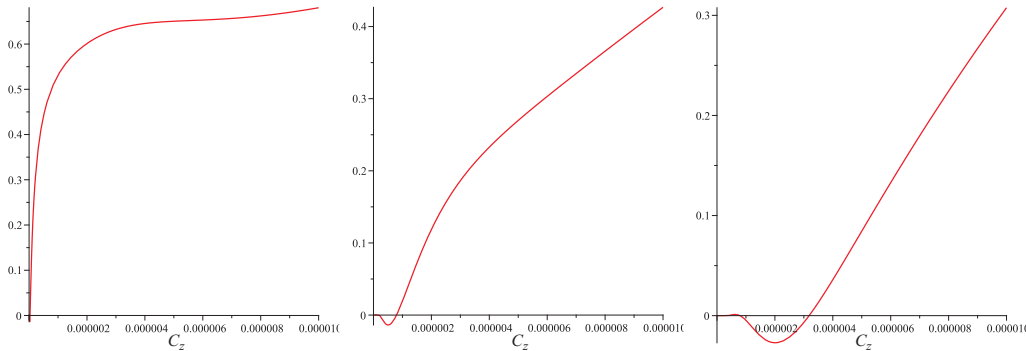


Fig. 8. Phase  $\psi(\xi)$  as a function of  $C_z$ , (left)  $\xi = 0.1$  ( $z = 0.1H$ ), (middle)  $\xi = 0.5$  ( $z = 0.5H$ ), (right)  $\xi = 1$  ( $z = H$ )

may be simplified in order to derive an analytical expression for the phase shift. Moreover, the analytical solution under this simplification is in a good agreement with the numerical solution obtained via FEM. The phase shift is presented as a function of the vertical coordinate  $z$  (depth),  $C_z$  and  $d$  by Eq. (19). On the other hand  $\psi$  may be calculated by using Eq. (29).

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